Evaluation of the Euclidean distortion of a distance-regular graph using semidefinite programming method

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Let $X$ be a metric space and $F$ be an embedding from $X$ to the $\ell_2$-Hilbert space. The distortion of $F$ is defined by the product of the Lipschitz constant of $F$ and the Lipschitz constant of $F^{-1}$, and the Euclidean distortion of $X$ is defined by the infimum of distortion amongst the embedding of $X$. It is not easy to determine the Euclidean distortion.

There are many researches of the Euclidean distortions of finite graphs. Moreover, for distance-regular graphs, lower bounds for the Euclidean distortions are known. In this paper, using semidefinite programming method, we rediscover lower bounds of Linial, London and Rabinovich.

**Key words**: distance-regular graph; Euclidean distortion; semidefinite programming method.

## 1 Introduction

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and $F$ be a map from $(X, d_X)$ into $(Y, d_Y)$. For a positive real $C$, a map $F$ is called a $C$-bilipschitz embedding if there exists $r > 0$ such that

$$rd_X(x, y) \leq d_Y(F(x), F(y)) \leq rC d_X(x, y)$$

(1)

for any $x, y \in X$. A bilipschitz embedding is an embedding which is $C$-bilipschitz for some $C \in \mathbb{R}_{\geq 0}$. Remark that if $F$ is a bilipschitz embedding then $F$ is injection. The smallest constant $C$ for which there exists $r > 0$ such that (1) is satisfied is called the distortion of $F$. (It is easy to see that such smallest constant exists.) The infimum of distortions of bilipschitz embeddings of $X$ into $Y$ is denoted $c_Y(X)$.

The distortion of $F$ is equal to

$$\sup_{x \neq y \in X} \frac{d_Y(F(x), F(y))}{d_X(x, y)} \leq \inf_{x \neq y \in X} \frac{d_Y(F(x), F(y))}{d_X(x, y)}.$$

The numerator of the above is called the Lipschitz constant of $F$ and the reciprocal of the denominator is called the Lipschitz constant of $F^{-1}$. By the definition of distortion, the distortion of $F$ is at least 1 and if the distortion of $F$ is equal to 1 then $F$ is an isometry. Thus $c_Y(X) \geq 1$ and if $c_Y(X) = 1$ then $X$ and $Y$ are isometric.

When $Y = \ell_p$, we use the notation $c_p(\cdot) = c_p(\cdot)$ and call this number the $\ell_p$-distortion of $X$. The parameter $c_p(X)$ is called the Euclidean distortion of $X$. We can obtain a trivial upper bound for $c_p(\cdot)$ of a finite graph. For a finite graph $\Gamma = (X, E)$ with diameter $d$, let $F : \Gamma \rightarrow \mathbb{R}^X$, $x \mapsto e_x$ (standard representation) as $\ell_p^X$, then $\|F(x) - F(y)\|_p = 2^{1/p}(1 - \delta_{xy})$, where $\delta_{xy}$ is the Kronecker delta. Hence the distortion of $F$ is $d$. This implies $c_p(\Gamma) \leq d$.

Let $X$ be an $n$-points metric space. Bourgain\(^{(2)}\) showed that $c_2(X) = O(\log n)$. However, it is not easy to determine the exact value of the Euclidean distortion $c_2(X)$ of a given $n$-point metric space. We have few examples of finite metric spaces whose Euclidean distortion is exactly given. The list of the examples only includes Hamming graphs (due to Enflo\(^{(4)}\)), Johnson graphs and all strongly regular graphs (due to Valentin\(^{(7)}\)). Linial, London and Rabinovich\(^{(5)}\) gave an algorithm to find the Euclidean distortions of finite metric spaces, and Linial and Magen\(^{(6)}\) showed some properties of the optimal embedding for the Euclidean distortions. However it is not easy to determine the Euclidean distortion of given metric space.

Our aim is to find a “good” evaluation of the Euclidean distortion of a finite graph $\Gamma$ of diameter $d$. We already have the trivial evaluation:

$$1 \leq c_2(\Gamma) \leq d.$$

Our concern is sharper evaluation of $c_2(\Gamma)$. The results of Linial, London and Rabinovich\(^{(5)}\) also give some lower bounds for the Euclidean distortions. After their work, Valentin\(^{(7)}\) showed specific lower bounds for the Euclidean distortions of distance-regular graphs using the parameters of the graphs.

**Theorem 1.1** (Valentin\(^{(7)}\)). Let $\Gamma$ be a distance-regular graph with $d$, $(\theta_h)_{h=0}^d$ be the eigenvalues of $\Gamma$ and $(v_h)_{h=0}^d$ be the polynomials related to $\Gamma$. Then the Euclidean distortion $c_2(\Gamma)$ of $\Gamma$ have the following lower bound:

$$c_2(\Gamma)^2 \geq \frac{d^2 v_d(\theta_0)}{v_1(\theta_0)} \min_{1 \leq h \leq d} \frac{v_h(\theta_0) - v_1(\theta_h)}{v_h(\theta_0) - v_d(\theta_h)}.$$
In this paper we first rediscover the results of Linial, London and Rabinovich\(^{(6)}\) and Linial and Magen\(^{(6)}\) using semidefinite programming methods. Next we refine Valentin’s lower bound. Finally we give upper bounds for the Euclidean distortions of distance-regular graphs using the primitive idempotents of the graphs.

2 Preliminary

Throughout this paper, for a set \(X\), let \(\text{Mat}_\mathbb{R}(X)\) be the set of matrices of size \(|X|\) over \(\mathbb{R}\) such that the rows and the columns are indexed by \(X\), and \(\text{Sym}_\mathbb{R}(X)\) be the set of symmetric matrices in \(\text{Mat}_\mathbb{R}(X)\). For \(M \in \text{Mat}_\mathbb{R}(X)\) and \(x, y \in X\), \(M_{xy}\) denotes the \((x, y)\)-entry of \(M\).

2.1 Distance-regular graph

We consider only finite undirected graphs without loops or multiple edges. Let \(\Gamma = (X, E)\) be such a graph, where \(X\) and \(E\) are the vertex and edge sets. For two vertices \(x\) and \(y\), \(\partial\Gamma(x, y)\) denotes the length of the shortest path joining \(x\) and \(y\). The diameter of \(\Gamma\) is the maximal distance occurring in \(\Gamma\). A connected graph \(\Gamma\) with diameter \(d\) is distance-regular if each of the following numbers

\[
\begin{align*}
|\{z \in X \mid \partial\Gamma(x, z) = i - 1, \partial\Gamma(z, y) = 1\}| & \quad (2) \\
|\{z \in X \mid \partial\Gamma(x, z) = i, \partial\Gamma(z, y) = 1\}| & \quad (3) \\
|\{z \in X \mid \partial\Gamma(x, z) = i + 1, \partial\Gamma(z, y) = 1\}| & \quad (4)
\end{align*}
\]

\((i = 0, 1, \ldots, d)\) does not depend on the choice of \(x, y \in X\) with \(\partial\Gamma(x, y) = i\). The numbers \((2), (3)\) and \((4)\) are denoted by \(b_i, a_i\) and \(c_i\) respectively. We always assume that \(c_d = d_d = 0\). The constant \(b_0\) is called the valency of \(\Gamma\) and is denoted by \(k\). Remark that \(a_i + b_i + c_i = k\) for each \(i = 0, 1, \ldots, d\). The numbers \(a_i, b_i, c_i\) are called the intersection numbers and the array \(\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}\) is called the intersection array of \(\Gamma\).

Let \(\Gamma = (X, E)\) be a distance-regular graph with diameter \(d\) and \(|X| = n\). For \(i = 0, 1, \ldots, d\), let \(A_i\) be the matrix in \(\text{Mat}_\mathbb{R}(X)\) and the \((x, y)\)-entry is 1 whenever \(x\) and \(y\) are at distance \(i\) and 0 otherwise. We call \(A_i\) the \(i\)-th distance matrix of \(\Gamma\). We abbreviate \(A := A_1\) and call this the adjacency matrix of \(\Gamma\). We have the recurrence relation

\[
A_iA = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}.
\]

This recurrence relation implies that there exist polynomials \(v_i(\theta)\) of degree exactly \(i\) such that \(A_i = v_i(A)\). Note \(v_0(\theta) = 1\) and \(v_1(\theta) = \theta\). We find that \(A_0, A_1, \ldots, A_d\) form a basis for a commutative algebra \(\mathfrak{A} = \text{Span}_\mathbb{R}(A_i)_{i=0}^d \subseteq \text{Sym}_\mathbb{R}(X)\). We call \(\mathfrak{A}\) the Bose-Mesner algebra of \(\Gamma\). Since \(A_i = v_i(A)\), it turns out that \(A\) generates \(\mathfrak{A}\). By (3, p.43), \(\mathfrak{A}\) has a second basis \(E_0, E_1, \ldots, E_d\) of the primitive idempotents of \(\Gamma\), and \(A\) can be written as \(A = \sum_{\theta \in \mathfrak{A}} \theta E_\theta\), where \(\theta \) is the eigenvalue of \(A\) associated with \(E_\theta\). Remark that \(\{E_\theta\}_{\theta \in \mathfrak{A}}\) are positive semi-definite. We denote by \(m_\theta\) the multiplicity of \(\theta\). For an eigenvalue \(\theta = \theta_\) we will also write \(E_\theta\) instead of \(E_\).

**Remark 2.1.** Take \(M \in \mathfrak{A}\). Since \(\{E_\theta\}_{\theta \in \mathfrak{A}}\) are positive semi-definite, \(M\) is positive semi-definite if and only if there exist non-negative numbers \(a_0, a_1, \ldots, a_d\) such that \(M = \sum_{\theta \in \mathfrak{A}} a_\theta E_\theta\).

Since \(A_i = v_i(A)\) and \(E_\theta\) are idempotents, we have

\[
A_i = \sum_{h=0}^d v_i(\theta_h)E_h
\]

for \(i \in \{0, 1, \ldots, d\}\). Let \(P\) be the square matrix of size \(d + 1\) whose \((h, i)\)-entry is \(v_i(\theta_h)\), that is,

\[
P = \begin{pmatrix}
v_0(\theta_0) & v_1(\theta_0) & \cdots & v_d(\theta_0) \\
v_0(\theta_1) & v_1(\theta_1) & \cdots & v_d(\theta_1) \\
\vdots & \vdots & \ddots & \vdots \\
v_0(\theta_d) & v_1(\theta_d) & \cdots & v_d(\theta_d)
\end{pmatrix}.
\]

Then (5) implies that \(P\) is non-singular. Hence there exists a square matrix \(Q = (Q_h(i))_{h,i=0}^d\) of size \(d + 1\) such that

\[
E_h = \frac{1}{n} \sum_{i=0}^d Q_h(i)A_i,
\]

that is,

\[
\sum_{h=0}^d Q_h(k) v_i(\theta_h) = n\delta_{kl}
\]

for \(k, l \in \{0, 1, \ldots, d\}\). Is is known that

\[
Q_0(i) = 1 \quad (i \in \{0, 1, \ldots, d\}),
\]

and

\[
v_i(\theta_h) = \frac{Q_h(i)}{Q_h(0)}
\]

for each \(h, i \in \{0, 1, \ldots, d\}\). The reader is referred to Bannai–Itô\(^{(1)}\) for more properties of \(P\) and \(Q\).

**Remark 2.2.** For \(M = \sum_{h=0}^d a_h E_h\) and \(x, y \in X\) with \(\partial\Gamma(x, y) = i\), we have

\[
M_{xy} = \left( \sum_{h=0}^d a_h E_h \right)_{xy} = \left( \sum_{h=0}^d a_h \frac{1}{n} \sum_{i=0}^d Q_h(i)A_i \right)_{xy} = \frac{1}{n} \sum_{h=0}^d a_h Q_h(i).
\]
2.2 Embedding of graphs onto spheres

Definition 2.3. Let \((X, d_X)\) be a finite metric space and \((Y, d_Y)\) be a metric space. We say that an embedding \(F: X \to Y\) is faithful if for every two pairs \((x, y), (x', y') \in X \times X\) we have

\[
d_Y(F(x), F(y)) = d_Y(F(x'), F(y'))
\]

whenever \(d_X(x, y) = d_X(x', y')\).

Suppose \((X, d_X) = (\Gamma, \partial \Gamma)\) and \(F\) is faithful, then we write \(d_Y(F(x), F(y)) = d_Y(i)\) whenever \(\partial \Gamma(x, y) = i\).

Definition 2.4. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, and let \(F\) be an embedding from \(X\) onto \(Y\).

1. A pair \((x, y) \in X \times X\) is called expanded if \(d_Y(F(x), F(y))/d_X(x, y)\) is maximal among all pairs in \(X \times X\).
2. A pair \((x, y) \in X \times X\) is called contracted if \(d_Y(F(x), F(y))/d_X(x, y)\) is minimal among all pairs in \(X \times X\).

Moreover suppose \((X, d_X) = (\Gamma, \partial \Gamma)\) and \(F\) is faithful.

1. A distance \(i\) is called expanded if there exists an expanded pair \((x, y)\) such that \(\partial \Gamma(x, y) = i\).
2. A distance \(i\) is called contracted if there exists a contracted pair \((x, y)\) such that \(\partial \Gamma(x, y) = i\).

Lemma 2.5. Let \(\Gamma = (X, E)\) be a graph of diameter \(d\), and \(F\) be a faithful embedding \(\Gamma\) onto a sphere \(S^N\). Let \(\partial \) be the distance among elements of \(F(\Gamma)\) on \(S^N\). Then the expanded distance is only 1, i.e., \(\partial(1)/\partial(k) = k\) for \(k \in \{2, 3, \ldots, d\}\).

Proof. For \(k \in \{2, 3, \ldots, d\}\), there exists a path \(x_0 \sim x_1 \sim x_2 \sim \cdots \sim x_k\) such that \(\partial(x_0, x_k) = k\) and \(\partial(x_{i-1}, x_i) = 1\) for \(i \in \{1, 2, \ldots, k\}\). By the triangle inequality, we have

\[
\partial(k) = \partial(F(x_0), F(x_k)) \\
\leq \sum_{i=1}^{k} \partial(F(x_{i-1}), F(x_i)) = \sum_{i=1}^{k} \partial(1) = k\partial(1),
\]

i.e., \(\partial(1)/\partial(k) = k\) follows.

Assume \(\partial(1)/\partial(k) = k\) follows, then

\[
\partial(F(x_0), F(x_k)) = \sum_{i=1}^{k} \partial(F(x_{i-1}), F(x_i))
\]

holds. This implies that \(F(x_0), F(x_{k-1}), F(x_k)\) lie on a line. However, this contradicts that \(F(x_0), F(x_{k-1}), F(x_k)\) lie on \(S^N\). Hence we get \(\partial(1)/\partial(k) = k\). \(\square\)

For a finite subset \(Z\) in a Euclidean space, the Gram matrix \(G\) of \(Z\) in \(\text{Sym}_g(Z)\) is defined by

\[
G_{xy} = x \cdot y \text{ for } x, y \in Z, \text{ where } \cdot \text{ is the standard inner product of the Euclidean space.}
\]

Lemma 2.6. Let \(\Gamma = (X, E)\) be a graph. For a semi-definite matrix \(M\) in \(\text{Sym}_g(X)\), there exists an embedding of \(\Gamma\) into the Euclidean space of dimension \(\text{Rank} M\).

Proof. Since \(M\) is semi-definite, there exists \((\text{Rank} M \times |X|)\)-matrix \(B\) such that \(B^T B = M\). Then we can regard the column vectors of \(B\) as vectors in \(\mathbb{R}^{\text{Rank} M}\). \(\square\)

Let \(\Gamma = (X, E)\) be a distance-regular graph with diameter \(d\), \(\{\partial(i)\}_{i=0}^{d}\) be the eigenvalues of \(\Gamma\) and \(\{E_h\}_{i=0}^{d}\) be the primitive idempotents of \(\Gamma\). For \(E_h\), put \(m = \text{Rank} E_h\). We define the map \(F_h\) from \(X\) into a Euclidean space such that, for \(x \in X\), \(F_h(x)\) is the \(x\)-th column of \(E_h\). Since \(E_h\) is an idempotent, \(E_h^2 E_h = E_h\) holds. This implies the Gram matrix of \(F_h(\Gamma)\) is \(E_h\). By Lemma 2.6, \(F_h\) is embedding of \(\Gamma\) into \(\mathbb{R}^m\). Moreover

\[
F_h(x) \cdot F_h(y) = (E_h)_{xy} = \frac{1}{n} \partial(i)
\]

for \(x, y \in X\) with \(\partial(x, y) = i\). Note that for each \(x \in X\), we have \(\|F_h(x)\|^2 = Q_h(0)/n\), hence \(F_h(\Gamma)\) is on a sphere \(S^{m-1}\). On the other hand we have

\[
\|F_h(x), F_h(y)\|^2 = (Q_h(0) - Q_h(i))/n
\]

for \(x, y \in X\) with \(\partial(x, y) = i\), hence \(F_h\) is faithful. By Lemma 2.5, the expanded distance is only 1. Therefore we get the following

Lemma 2.7. \(F_h\) is embedding of \(\Gamma\) onto \(S^{m-1}\) such that the expanded distance is only 1.

3 Semi-definite programming

For a finite graph \(\Gamma = (X, E)\) with the path distance \(\partial\), let \(\tilde{\mathcal{C}}\) be the set of \((M, r^2)\) satisfying \((M, r^2) \neq (0, 0), M \in \text{Sym}_g(X)\) is positive semi-definite, \(r \geq 0\) and

\[
M_{xx} + M_{yy} - 2M_{xy} - r^2 \partial(x, y)^2 \geq 0
\]

for \(x, y \in X\) \((x \neq y)\) and, for a non-negative real \(D\), let \(\tilde{S}_D\) be the set of \((M, r^2)\) satisfying \((M, r^2) \neq (0, 0), M \in \text{Sym}_g(X)\) and

\[
-M_{xx} - M_{yy} + 2M_{xy} + r^2 D \partial(x, y)^2 \geq 0
\]

for \(x, y \in X\) \((x \neq y)\). Then \(\tilde{\mathcal{C}}\) and \(\tilde{S}_D\) are cones in \(\text{Sym}_g(X) \oplus \mathbb{R}\).
Lemma 3.1. There exists a $\sqrt{D}$-bilipschitz embedding of $\Gamma$ into $\ell_2$ if and only if $C \cap S_D \neq \emptyset$.

Proof. Assume $F$ is a $\sqrt{D}$-bilipschitz embedding of $\Gamma$ into $\ell_2$. Let $G$ be the Gram matrix of $F(X)$. Then $\|F(x) - F(y)\|^2 = G_{xx} + G_{yy} - 2G_{xy}$. By the definition of bilipschitz, we can check $(G, r^2) \in C$ and $(G, r^2) \in S_D$ for some $r > 0$. Hence this implies $C \cap S_D \neq \emptyset$.

Assume $(M, r^2) \in C \cap S_D$. By Lemma 2.6, there exists an embedding $F$ of $\Gamma$ into the Euclidean space with

$$\|F(x) - F(y)\|^2 = M_{xx} + M_{yy} - 2M_{xy}. \quad (10)$$

Then $F$ satisfies the condition of $\sqrt{D}$-bilipschitz.

Lemma 3.1 implies that $C$ is the Euclidean distortion of $\Gamma$ if and only if $C = \inf \{\sqrt{D} : D > 0 \text{ with } C \cap S_D \neq \emptyset\}$. The study of the intersection of two cones $C$ and $S_D$ using convex analysis is called positive definite programming method. If $\Gamma$ is distance-regular, the two cones $C$ and $S_D$ became more simple.

Theorem 3.2 (Valentin’s). Let $\Gamma = (X, E)$ be a distance-regular graph. Then, there exists a faithful embedding of $\Gamma$ into Euclidean space with minimal distortion. In particular, this embedding is in a sphere centered at the origin in the Euclidean space.

Theorem 3.2 yields that for $\Gamma$ is distance-regular, an optimal solution $(M, r^2) \in C \cap S_D$ of $\inf \{\sqrt{D} : D > 0 \text{ with } C \cap S_D \neq \emptyset\}$ can be found in $\mathbb{R}^d$. By Remarks 2.1 and 2.2, $C := C \cap (\mathbb{R} \oplus \mathbb{R})$ can be regarded as a cone

$$\left\{(a_0, \ldots, a_d, R) \mid \begin{array}{l}
a_h \geq 0 \quad (0 \leq h \leq d), \\
R \geq 0, \\
\frac{2}{n} \sum_{h=0}^d a_h (Q_h(0) - Q_h(i)) \\
- R^2 + \sum_{h=0}^d a_h (Q_h(0) - Q_h(i)) \geq 0 \quad (1 \leq i \leq d), \\
(a_0, \ldots, a_d, R) \neq 0
d\end{array}\right\} \quad (11)$$

in $\mathbb{R}^{d+2}$ and also $S_D := S_D \cap (\mathbb{R} \oplus \mathbb{R})$ can be regarded as a cone

$$\left\{(a_0, \ldots, a_d, R) \mid \begin{array}{l}
a_h \in \mathbb{R} \quad (0 \leq h \leq d), \\
R \in \mathbb{R}, \\
\frac{2}{n} \sum_{h=0}^d a_h (Q_h(0) - Q_h(i)) \\
R^2 + \sum_{h=0}^d a_h (Q_h(0) - Q_h(i)) \geq 0 \quad (1 \leq i \leq d), \\
(a_0, \ldots, a_d, R) \neq 0
\end{array}\right\} \quad (12)$$

Henceforth, we write $(a, R)$ or $(a_0, \ldots, a_d, R)$ instead of $(a_0, \ldots, a_d, R)$. For $(a, R), (a', R') \in \mathbb{R}^{d+2}$, the standard inner product is given by $(a, R) \cdot (a', R') = \sum_{h=0}^d a_h a'_h + RR'$. The dual cones of $C$ and $S_D$ are defined by

$$C^* = \{(b, s) \in \mathbb{R}^{d+2} \mid (a, r) \cdot (b, s) \geq 0, \quad (a, r) \in C\} \quad (13)$$

and

$$S_D^* = \{(b, s) \in \mathbb{R}^{d+2} \mid (a, r) \cdot (b, s) \geq 0, \quad (a, r) \in S_D\},$$

respectively. Then $C^*$ is written in

$$\left\{(b, s) \mid b_h \geq 0 \quad (0 \leq h \leq d), \quad s \geq 0\right\}$$

$$+ \left\{\sum_{i=1}^d \alpha_i \left(\frac{2}{n} (Q_h(0) - Q_h(i))\right) \left(-R^2 + \sum_{h=0}^d a_h (Q_h(0) - Q_h(i)) \geq 0 \right) \right\} \quad (11)$$

and also $S_D^*$ is

$$\left\{\sum_{i=1}^d \beta_i \left(\frac{2}{n} (Q_h(0) - Q_h(i))\right) \left(-R^2 + \sum_{h=0}^d a_h (Q_h(0) - Q_h(i)) \geq 0 \right) \right\} \quad (12)$$

Lemma 3.3. For two nonnegative reals $D$ and $D'$, if $D \leq D'$, then $S_D \subset S_{D'}$.

Lemma 3.4. If $C \cap S_D \neq \emptyset$, then $C^* \cap (-S_D^*) = \emptyset$, where $(-S_D^*)$ is the interior of $-S_D^*$.

Proof. Assume $C^* \cap (-S_D^*) = \emptyset$. Let $(a, R) \in C \cap S_D$ and $(b, s) \in C^* \cap (-S_D^*)$. Since $(a, R) \in C$ and $(b, s) \in C^*$, we have $(a, R) \cdot (b, s) \geq 0$. On the other hand, by $(a, R) \in S_D$ and $(b, s) \in (-S_D^*)$, we have $(a, R) \cdot (b, s) < 0$. It is contradiction.

Theorem 3.5 (SDP method). If $D \in \mathbb{R}$ satisfies $C^* \cap (-S_D^*) \neq \emptyset$, then $D \leq c_2(\Gamma)^2$ holds.

Proof. By Lemmas 3.1 and 3.3, we have $C \cap S_D \neq \emptyset$ for any $C$ with $C \geq c_2(\Gamma)^2$. This implies if $C \cap S_D = \emptyset$, then $D < c_2(\Gamma)^2$. By Lemma 3.4, if $C^* \cap (-S_D^*) \neq \emptyset$, then $D < c_2(\Gamma)$. Moreover if $C^* \cap (-S_D^*) = \emptyset$, then $D \leq c_2(\Gamma)$ also.

Using Theorem 3.5, we find the lower bound for $c_2(\Gamma)$. Fix nonnegative reals $b = (b_0, b_1, \ldots, b_d)$. Assume $C^* \cap (-S_D^*) \neq \emptyset$, then there exist positive reals $\alpha_i$, $\beta_i$ and $s$ such that

$$b_h = \frac{2}{n} \sum_{i=1}^d (\beta_i - \alpha_i) (Q_h(0) - Q_h(i)) \quad (13)$$

for $h \in \{0, 1, \ldots, d\}$ and

$$s - \sum_{i=1}^d \alpha_i t^2 = - \sum_{i=1}^d \beta_i t^2 D \quad (14)$$

by (11) and (12).

Remark 3.6. Under the assumption, we have $b_0 = 0$. Because $b_0 = \frac{2}{n} \sum_{i=1}^d (\beta_i - \alpha_i) (Q_0(0) - Q_0(i)) = 0$ by (7).
Let $\hat{Q} = (Q_h(0) - Q_h(i))^d_{i,h=1}$ and $\hat{P} = -\frac{1}{n} (v_l(\theta_h))^d_{h,l=1}$.

**Lemma 3.7.** $\hat{Q}^{-1} = \hat{P}$.

**Proof.** (6) and (7) imply that, for $k, l \in \{1, 2, \ldots, d\}$,

$$(\hat{Q} \hat{P})_{k,l} = -\frac{1}{n} \sum_{h=1}^{d} (Q_h(0) - Q_h(k)) v_l(\theta_h)$$

$$= -\frac{1}{n} \sum_{h=1}^{d} Q_h(0) v_l(\theta_h) + \frac{1}{n} \sum_{h=1}^{d} Q_h(k) v_l(\theta_h)$$

$$= -\frac{1}{n} (0 - Q_h(0)) v_l(\theta_0)$$

$$+ \frac{1}{n} (n \delta_{k,l} - Q_h(k) v_l(\theta_0))$$

$$= -\frac{1}{n} (0 - Q_h(0)) v_l(\theta_0) + \frac{1}{n} v_l(\theta_0)$$

$$= \delta_{k,l}.$$

Hence the desired result holds. □

By (13), we have

$$(b_h)_{h=1}^d = \frac{2}{n} (\beta_i - \alpha_i)^d_{i=1} \hat{Q}.$$

By Lemma 3.7, we have

$$(\beta_i - \alpha_i)^d_{i=1} = \frac{n}{2} (b_h)_{h=1}^d \hat{P}.$$

Hence, for $i \in \{1, 2, \ldots, d\}$,

$$\beta_i - \alpha_i = -\frac{1}{2} \sum_{h=1}^{d} b_h v_l(\theta_h).$$

Since $\alpha_i = \beta_i + \frac{1}{2} \sum_{h=1}^{d} b_h v_l(\theta_h) \geq 0$ and $\beta_i \geq 0$, we have

$$\beta_i \geq \max\{0, \frac{1}{2} \sum_{h=1}^{d} b_h v_l(\theta_h)\}. \quad (15)$$

Put $\gamma_i := \frac{1}{2} \sum_{h=1}^{d} b_h v_l(\theta_h)$ and $\gamma_i := \{i | \gamma_i > 0\}$.

By (14), we have

$$D = \frac{\sum_{i=1}^{d} \alpha_i \gamma_i^2}{\sum_{i=1}^{d} \beta_i \gamma_i^2}.$$ 

Then

$$D = \frac{\sum_{i=1}^{d} (\beta_i + \gamma_i) \gamma_i^2}{\sum_{i=1}^{d} \beta_i \gamma_i^2} - \frac{1}{\sum_{i=1}^{d} \gamma_i \gamma_i^2} - \frac{1}{\sum_{i=1}^{d} \gamma_i \gamma_i^2}.$$  

In this situation, if we take

$$\beta_i = \begin{cases} -\gamma_i & \text{if } \gamma_i < 0 \\ 0 & \text{if } \gamma_i \geq 0 \end{cases}$$

then $D$ is maximized for fix numbers $b$, and the value $D$ is

$$-\frac{\sum_{i \in \gamma^+} \gamma_i \gamma_i^2}{\sum_{i \in \gamma^-} \gamma_i \gamma_i^2}.$$

Put

$$\Delta(b) := -\frac{\sum_{i \in \gamma^+} \gamma_i \gamma_i^2}{\sum_{i \in \gamma^-} \gamma_i \gamma_i^2}.$$

**Theorem 3.8.**

$$c_2(\Gamma)^2 \geq \sup \{ \Delta(b) | b_0 = 0, b_1, b_2, \ldots, b_d \geq 0 \}.$$

For nonnegative numbers $b$ and positive real $c$, we can check $\Delta(b) = \Delta(c, b)$. Therefore there exists $b' \in [0, 1]^{d+1}$ such that $b'$ attain sup $\{ \Delta(b) | b_0 = 0, b_1, b_2, \ldots, b_d \geq 0 \}$.

**Theorem 3.9 (Linial, London and Rabinovich(0)).**

$$c_2(\Gamma)^2 \geq \max \{ \Delta(b) | b_0 = 0, b_1, b_2, \ldots, b_d \in [0, 1] \}.$$

**Lemma 3.10.** For any $(a, R) \in C \cap S_P$ and $(b, s) \in C^* \cap (\gamma^c \cap (-S_P^c))$, we have $\sum_{h=1}^{d} a_h b_h + Rs = 0$.

**Proof.** Since $(a, R) \in C$ and $(b, s) \in C^*$, we have $(a, R) \cdot (b, s) \geq 0$. This implies $\sum_{h=1}^{d} a_h b_h + Rs \geq 0$. On the other side, since $(a, R) \in S_P$ and $(b, s) \in -S_P^c$, we have $(a, R) \cdot (b, s) \leq 0$. This implies $\sum_{h=1}^{d} a_h b_h + Rs \leq 0$. Hence we get $\sum_{h=1}^{d} a_h b_h + Rs = 0$. □

**Theorem 3.11 (Linial and Magen(0)).** Assume $b$ satisfies $\Delta(b) = c_2(\Gamma)^2$ and $(a, R) \in C \cap S_{c_2(\Gamma)^2}$. Put $\overline{D}(i) = \frac{2}{n} \sum_{h=0}^{d} a_h (Q_h(0) - Q_h(i))$. Then following hold.

(i) $\sum_{h=0}^{d} a_h b_h = 0$.

(ii) For $i \in \gamma^+$, the distance $i$ is contracted, i.e., $\overline{D}(i) = \min_{1 \leq k \leq d} \overline{D}(k)$.

(iii) For $i \in \gamma^-$, the distance $i$ is expanded, i.e., $\overline{D}(i) = \max_{1 \leq k \leq d} \overline{D}(k)$.

**Proof.** The element of $C^* \cap (\gamma^c \cap (-S_{c_2(\Gamma)^2})$ related to $b$ forms

$$(b_h + \sum_{i \in \gamma^+} \gamma_i \frac{2}{n} (Q_h(0) - Q_h(i)))_{h=0}^{d}, -\sum_{i \in \gamma^+} \gamma_i \gamma_i^2 \quad (16)$$

or

$$\left(\sum_{i \in \gamma^+} \gamma_i \frac{2}{n} (Q_h(0) - Q_h(i))_{h=0}^{d}, -c_2(\Gamma)^2 \sum_{i \in \gamma^+} \gamma_i \gamma_i^2 \right). \quad (17)$$
\[(i)\text{ and } (ii)\] Apply the relation \((a, R)\) and (16) to Lemma 3.10, we get

\[
\sum_{h=0}^{d} a_h \left( b_h + \sum_{i \in \gamma^+} \gamma_i^2 (Q_h(0) - Q_h(i)) \right) - R \sum_{i \in \gamma^+} \gamma_i^2 = 0.
\]

is vanish, i.e.,

\[
\sum_{h=0}^{d} a_h b_h + \sum_{i \in \gamma^+} \gamma_i^2 (Q_h(0) - Q_h(i)) - R \sum_{i \in \gamma^+} \gamma_i^2 = 0.
\]

Since \(R = \min_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2}\), we have

\[
\sum_{h=0}^{d} a_h b_h + \sum_{i \in \gamma^+} \gamma_i^2 \left( \frac{\partial^2(i)}{i^2} - \min_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2} \right) = 0.
\]

Since \(\sum_{h=0}^{d} a_h b_h \geq 0\), \(\gamma_i^2 > 0\) for \(i\) with \(\gamma_i > 0\), and \(\frac{\partial^2(i)}{i^2} - \min_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2} \geq 0\), we have \(\sum_{h=0}^{d} a_h b_h = 0\) and \(\frac{\partial^2(i)}{i^2} = \min_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2}\) for \(i \in \gamma^+\).

\[\text{[iii]}\] Apply the relation \((a, R)\) and (17) to Lemma 3.10, we get

\[
\sum_{h=0}^{d} a_h \left( \sum_{i \in \gamma^-} \gamma_i^2 (Q_h(0) - Q_h(i)) \right) - c_2(\Gamma)^2 R \sum_{i \in \gamma^-} \gamma_i^2 = 0,
\]

is vanish, i.e.,

\[
\sum_{i \in \gamma^-} \gamma_i \frac{\partial^2(i)}{i^2} - R c_2(\Gamma)^2 \sum_{i \in \gamma^-} \gamma_i^2 = 0.
\]

Since \(R c_2(\Gamma)^2 = \max_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2}\), we have

\[
\sum_{i \in \gamma^-} \gamma_i^2 \left( \frac{\partial^2(i)}{i^2} - \max_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2} \right) = 0.
\]

Since \(\gamma_i^2 < 0\) for \(i \in \gamma^-\), and \(\frac{\partial^2(i)}{i^2} \leq \max_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2}\), we have \(\frac{\partial^2(i)}{i^2} = \max_{1 \leq k \leq d} \frac{\partial^2(k)}{k^2}\) for \(i \in \gamma^-\).

\[\square\]

4 Upper and lower bound for \(c_2(\Gamma)\)

In this section, we give bounds for the Euclidean distortion of a distance-regular graph \(\Gamma\).

4.1 Lower bound for \(c_2(\Gamma)\)

For \(l \in \{2, 3, \ldots, d\}\), let

\[
m(l) := \min_{1 \leq j \leq d, l \leq j} \frac{Q_j(0) - Q_j(1)}{Q_j(0) - Q_j(l)}.
\]

and

\[
b^{(l)}_h = (Q_h(0) - Q_h(1)) - m(l)(Q_h(0) - Q_h(l))
\]

\((1 \leq h \leq d)\). By the definition, \(b^{(l)}_h \geq 0\) for \(1 \leq h \leq d\). Then

\[
\gamma_i = \frac{1}{2} \sum_{h=1}^{d} b^{(l)}_h v_i(\theta_h)
\]

\[
= \frac{1}{2} \sum_{h=1}^{d} (Q_h(0) - Q_h(1)) v_i(\theta_h) - m(l)(Q_h(0) - Q_h(l)) v_i(\theta_h)
\]

\[
= \frac{\gamma_i}{2} (\delta_{l, i} - m(l) \delta_{l, i}),
\]

i.e., \(\gamma_i = \frac{\gamma_i}{2} < \gamma_i = \frac{\gamma_i}{2} m(l) > 0\) and \(\gamma_i = 0\) for \(i \neq 1, l\). Hence we have \(\Delta((b^{(l)}_h)^2) = l^2 m(l)\).

Moreover we have the following result.

Theorem 4.1.

\[
c_2(\Gamma)^2 \geq \max_{2 \leq l \leq d} \min_{1 \leq j \leq d, l \leq j} \frac{l^2 Q_j(0) - Q_j(1)}{Q_j(0) - Q_j(l)}.
\]

By (8), Theorem 4.1 can be expressed by

\[
c_2(\Gamma)^2 \geq \max_{2 \leq l \leq d} \min_{1 \leq j \leq d, l \leq j} \frac{l^2 v_i(\theta_0) v_i(\theta_0) - v_i(\theta_l)}{v_i(\theta_0) - v_i(\theta_l)}.
\]

4.2 Upper bound for \(c_2(\Gamma)\)

We consider the embedding \(F_j\) in terms of \(E_j\). By Lemma 2.7, the distortion of \(F_j\) is

\[
\frac{Q_j(0) - Q_j(1)}{\min_{2 \leq l \leq d} \frac{Q_j(0) - Q_j(l)}{l^2}} \leq \frac{l^2 Q_j(0) - Q_j(1)}{Q_j(0) - Q_j(l)}.
\]

Therefore we have the following result.

Theorem 4.2.

\[
c_2(\Gamma)^2 \leq \max_{1 \leq j \leq d} \min_{2 \leq l \leq d} \frac{l^2 Q_j(0) - Q_j(1)}{Q_j(0) - Q_j(l)}.
\]

By (8), Theorem 4.2 can be expressed by

\[
c_2(\Gamma)^2 \leq \max_{1 \leq j \leq d} \min_{2 \leq l \leq d} \frac{l^2 v_i(\theta_0) v_i(\theta_0) - v_i(\theta_l)}{v_i(\theta_0) - v_i(\theta_l)}.
\]

Remark 4.3. Every distance-regular graph which we have already checked satisfies \(j = 1\). Moreover almost distance-regular graphs satisfy \(l = d\). However there exist some counterexamples of \(l = d\). A distance-regular graph with

\[
\{22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22\}
\]

is such an example.
文 献


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